A Stochastic Approach to the Construction of One-Dimensional Chaotic Maps with Prescribed Statistical Properties

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Abstract

We use a recently found parametrization of the solutions of the inverse Frobenius-Perron problem within the class of complete unimodal maps to develop a Monte-Carlo approach for the construction of one-dimensional chaotic dynamical laws with given statistical properties, i.e. invariant density and autocorrelation function. A variety of different examples are presented to demonstrate the power of our method.

1   Introduction

Chaotic dynamical systems show a rich diversity of possible behaviour with respect to their statistical properties. In recent years much work has been devoted to the understanding of how these statistical properties emerge from the dynamics. The most popular area for such investigations are unimodal 1-d maps [1, 2, 3, 4]. They became widespread due to two main advantages: their dynamics can be calculated efficiently enough to make extensive numerical investigations and the variety of statistical properties within this class of dynamical laws is

very large. Certain features of special systems can even be investigated analytically. The interplay between the dynamical and statistical behaviour becomes more transparent if we consider the inverse problem, i.e. given the statistical behaviour of a system how could we extract relevant information on the possible dynamics. This subject has recently been addressed and discussed in the literature [5, 6, 7, 8, 9, 10] and from a communications point of view in refs. [11, 12]. The purpose is then to construct an one-dimensional fully chaotic (and ergodic) dynamical law for a given invariant density or/and time autocorrelation function. Given the latter we can on the one hand calculate all expectation values, i.e. statistical averages, of measurable observables depending exclusively on the single dynamical variable. On the other hand the given autocorrelation function provides us with valuable information on the dynamics of the trajectories of the system. It represents an important quantity strongly related to the physics described by the dynamical system. The construction of a dynamical system with prescribed statistical properties can have the background of either some experimentally given data for which a dynamical system should be modelled or it can be justified by the fact that a definite behaviour of the correlation function is required in the context of the control of a physical system [9, 10].

Restricting ourselves to the first part of the problem, i.e. the construction of a map with a given invariant density (the so called inverse Frobenius-Perron problem (IFPP)) we have recently found a general and for practical purposes very helpful representation of the solution of the IFPP within the class of smooth complete and unimodal maps [7]. However the combined problem (both the invariant density and time autocorrelation function are given) is much more complicated. Some progress in this direction can be made if one uses the class of piecewise linear Markov maps as a basis for this search. Constructing such a map with a given autocorrelation function one can then obtain the proper invariant density performing a suitable conjugation (coordinate) transformation [8]. A substantial disadvantage of this method is that the changes of the autocorrelation function in the course of the conjugation transformation are not fully under control. Another rather anaesthetic aspect is the fact that in many cases the solution found to the inverse problem does not fulfill certain smoothness criteria.

The purpose of this letter is to present a new approach to the inverse problem including the autocorrelation function. It is based on the general representation of solutions of the IFPP found in [7] for complete smooth unimodal maps. We suitably parametrize the key function h_f occurring in this approach and develop a stochastic (Monte-Carlo) algorithm to determine the optimal solution such that the χ^2 deviation of the resulting autocorrelation from a desired function is minimized. The paper is organized as follows: In section 2 we briefly review the representation of our solution to the IFPP. In section 3 we present the Monte-Carlo algorithm used to perform the χ^2 minimization with respect to the time auto-

correlation function. Finally in section 4 we provide several applications in the framework of our Monte-Carlo scheme and discuss the convergence properties of our approach.

2   The general solution of the IFPP for smooth uni-modal 1-d maps

In a recent paper [7] we investigated the problem of designing dynamical systems which possess an arbitrary but fixed invariant density. The solution to this inverse problem is of great interest for the numerical simulation of real physical systems as well as for the understanding of the relationship between the functional form of the map and the statistical features of the corresponding dynamics. We were able to derive a general representation of all ergodic and chaotic complete unimodal maps with a given invariant density. This corresponds to the general solution of the so called inverse Frobenius-Perron problem [2, 5, 8, 13, 14, 15, 16] within this class of maps. We review here the basic aspects of our approach.

As a starting-equation for the construction of the map we use the Frobenius-Perron equation:

$$\rho(y)\,|dy| = \sum_{x_i=f^{-1}(y)} \rho(x_i)|dx_i| \tag{1}$$

where the summation runs over all preimages of y (for unimodal maps $i = L(left), R(right)$). For any given complete unimodal $f(x)$ the right preimage of y is determined if the left preimage is given and vice versa. The essential feature of our approach is the following: A prescribed relation between the two preimages reduces the number of independent differentials on the rhs of eq.(1) and allows to integrate the Frobenius-Perron equation. Such a relation is given by the function $h_f(x)$ which maps the position of the left preimage onto the position of the right one:

$$
\begin{aligned}
h_f &: \quad [0, x_{\max}] \longrightarrow [x_{\max}, 1] \\
x_R &= h_f(x_L) \quad \text{ with } f(x_L) = f(x_R)
\end{aligned}
\tag{2}
$$

where $x_{\max}$ is the position of the maximum of the map. $h_f(x)$ is a monotonously decreasing function on the defining interval and it is differentiable with the exception of a finite number of points. It obeys the equations:

$$
\begin{aligned}
h'_f(x) &< 0 \qquad x \in [0, x_{\max}] \\
h_f(0) &= 1 \qquad h_f(x_{\max}) = x_{\max}
\end{aligned}
\tag{3}
$$

In terms of h_f we obtain the general solution to IFPP within the class of unimodal maps as:

$$f(x) = \mu^{-1} \left(1 - |\mu(x) - \mu(H_f(x))| \right) \tag{4}$$

where $H_f(x)$ is:

$$H_f(x) = \begin{cases} h_f(x) & ; \quad 0 \leq x < x_{\max} \\ h_f^{-1}(x) & ; \quad x_{\max} \leq x \leq 1 \end{cases} \tag{5}$$

and $\mu(x) = \int_0^x \rho(y)dy$ is the corresponding invariant measure. All unimodal maps with prescribed invariant density $\rho(x)$ are given by (4), $h_f(x)$ taking on all possible functional forms obeying (2) and (3) (for more details on the above derivation as well as its relation to the standard conjugation procedure we refer the reader to [7]).

The above shows that fixing the invariant measure is a relatively weak constraint in the framework of the inverse problem and that there is still a considerable freedom to model the mapping. In the next section we will use this freedom and suitably parametrize $h_f(x)$ such that, by tuning appropriately the corresponding parameters, we get a map with an autocorrelation function possessing a minimum deviation (to be defined below) from a desired correlation function.

3   Stochastic optimization as a tool for the construction of a map

Before presenting our search method for the optimal map in the sense of some desired statistical properties let us specify in more detail what we want to achieve and what our input for the problem at hand is. Consider a given invariant density $\rho(x)$ (corresponding to a finite measure) which arises in the asymptotic limit $(t \to \infty)$ from the dynamics of some unknown $1 - d$ fully chaotic single humped map. Consider also as given the first m values $\{C(1), C(2), .., C(m)\}$ of the time autocorrelation function $C(n)$ (note that $C(0)$ is determined entirely through $\rho(x)$). We are seeking a map $f(x)$ which possesses the above statistical properties $(\rho(x), C(n)(n = 0, ..., m))$. We focus in the following on the representation (4) for all admissible maps where the auxiliary functions $h_f(x)$ fulfill the requirements (2,3). Using this expression for $f(x)$ it is guaranteed that the map we are looking for possesses the invariant density $\rho(x)$ (provided of course that $\mu(x) = \int_0^x \rho(t) \, dt$ in (4) exists). All our freedom in modelling the map $f(x)$ is now contained in the auxiliary function $h_f(x)$. We will use a suitable representation of $h_f(x)$ to parametrize the map $f(x)$. This parametrization is the starting point for a stochastic optimization procedure to obtain a map with the desired autocorrelation function $C(n)$ at times $n = 1, 2, .., m$. One of the simplest ansatz would be

to write down a piecewise linear expression for h_f. Our knowledge of the properties and dynamics of $1 - d$ maps however suggests that the local behaviour of the map in the neighbourhood of special points, like for example a marginal unstable fixed point in the case of intermittent dynamics (see ref.[17] and references therein), plays a crucial role in determining the statistical properties of the map. A more intuitive ansatz is therefore needed. Here we will proceed as follows.

We define a $1 - d$ lattice with $N + 1$ points in the interval $[0, x_{\max}]$. The coordinates of the lattice points are $x_p(i)$, $i = 0, 1, .., N$ with $x_p(0) = 0$ and $x_p(N) = x_{\max}$. The function $h_f(x)$ is piecewice defined. We will call in the following the expression of h_f in the $i - th$ interval $[x_p(i - 1), x_p(i))$ an **element** and use the notation $h_{f,i}$ for the $i - th$ element. In practice one is free to choose many different expressions for the i-th element. Constraints on h_f as for example continuity in $[0, x_{\max}]$ together with the conditions (2,3) however restrict the possible forms. A desirable but not necessary requirement is that $h_{f,i}$ should be analytically invertible in order to easily extend the solution in $[0, x_{\max}]$ to the interval $(x_{\max}, 1]$ (see (5)). Here we will investigate two different choices for the element $h_{f,i}$. In both cases continuity of h_f is guaranteed. The first case, refered to in the following as model I, is obtained by choosing the order of $h_{f,i}$ at the left point $x_p(i - 1)$ in the interval $[x_p(i - 1), x_p(i))$. This parametrization allows us to fit the derivative of the map at $x = 0$ and has the advantage of using a minimal set of parameters in describing the element $h_{f,i}$. The disadvantage of this choice is that one cannot fit independently the order of the maximum of the map at the right point $x_p(N)$ of the last interval $[x_p(N - 1), x_p(N)]$. This is the reason for considering also a second case (model II). Here we use for the element $h_{f,i}$ an expression representing expansions around both limiting points $x_p(i - 1), x_p(i)$ which match together at some point of the interval $[x_p(i - 1), x_p(i)]$. An additional parameter is therefore required in order to satisfy continuity for $h_{f,i}$ and its first derivative in $(x_p(i - 1), x_p(i))$. Due to this fact model II uses 2 more parameters than model I for the description of a single element $h_{f,i}$.

Let us now discuss the two models in more detail. For model I the element $h_{f,i}$ is given by:

$$
\begin{aligned}
h_{f,i} &= (y_p(i) - y_p(i - 1)) \left(\frac{x - x_p(i - 1)}{x_p(i) - x_p(i - 1)} \right)^{\alpha(i)} + y_p(i - 1) \\
x &\in [x_p(i - 1), x_p(i)) \; ; \quad i = 1, 2, .., N
\end{aligned}
\tag{6}
$$

where $y_p(i - 1), y_p(i)$ are the values of h_f at the points $x_p(i - 1), x_p(i)$ respectively, while the power $\alpha(i)$ determines the local behaviour of h_f around $x_p(i - 1)$. Due to the constraints (2,3) we get: $y_p(0) = 1$ and $y_p(N) = x_{\max}$. The monotony of h_f implies that $y_p(i-1) > y_p(i)$ for $i = 1, 2, .., N$. The above ansatz determines the map $f(x)$ in the interval $[0, x_{\max}]$. To find $f(x)$ in the remaining interval $[x_{\max}, 1]$ we take advantage of eq.(4) and therefore have

to invert the auxiliary function $h_f(x)$. It is straightforward to show that the expression (6) can be inverted analytically leading to a closed form for $h_f^{-1}(x)$. The above defined ansatz for h_f is in general piecewise smooth, i.e. smooth with the exception of a finite number of points.

In model II we use the following ansatz for the element $h_{f,i}$:

$$h_{f,i}(x) = y_p(i-1) - c_L(i)(x - x_p(i-1))^{\alpha_L(i)} \; ; \; x \in [x_p(i-1), x_s(i)) \tag{7}$$

$$h_{f,i}(x) = y_p(i) + c_R(i)(x_p(i) - x)^{\alpha_R(i)} \quad ; \quad x \in [x_s(i), x_p(i)] \tag{8}$$

where we have introduced four new parameters $c_L(i), c_R(i), \alpha_R(i), x_s(i)$ and a new point $x_s(i) \in [x_p(i-1), x_p(i))$ for each $i = 1, ..., N$. Two of these parameters, namely $c_L(i)$ and $c_R(i)$, can be fixed demanding continuity of h_f and its first derivative at $x = x_s(i)$. We arrive then at the following generating formula for h_f:

$$h_{f,i}(x) = y_p(i-1) - \quad S_i \quad \alpha_R(i)(x_s(i) - x_p(i-1))^{1-\alpha_L(i)}(x - x_p(i-1))^{\alpha_L(i)} \; ;$$
$$x \quad \in \quad [x_p(i-1), x_s(i))$$
$$h_{f,i}(x) = y_p(i) + \quad S_i \quad \alpha_L(i)(x_p(i) - x_s(i))^{1-\alpha_R(i)}(x_p(i) - x)^{\alpha_R(i)} \; ;$$
$$x \quad \in \quad [x_s(i), x_p(i)) \tag{9}$$

with:

$$S_i = \frac{(y_p(i-1) - y_p(i))}{\alpha_R(i)(x_s(i) - x_p(i-1)) + \alpha_L(i)(x_p(i) - x_s(i))} \tag{10}$$

$h_f(x)$ given in eq.(9) can also be inverted analytically to obtain h_f^{-1} defined in $[x_{\max}, 1]$. Comparing eqs.(6,9) we see immediately that the element of model II has 2 more parameters than the element of model I.

In the remaining part of this section we will discuss the algorithm which allows us to obtain dynamical systems possessing certain statistical properties based on the above ansatz for the auxiliary function $h_f(x)$. To this end we will concentrate on the parametrization of model I. One can then directly apply these ideas to the case of model II. We have determined the ansatz of $h_f(x)$ and therefore also of $f(x)$ in terms of the parameters $\{x_p(i), y_p(i), \alpha(i)\}$. Once the values of these parameters are chosen the map is completely specified. Due to its appearance (see eq.(4)) we automatically know its invariant density (measure) and expectation values of observables. The corresponding autocorrelation function can be obtained via its defining formula

$$C(n) = \int_0^1 x f^{(n)}(x) d\mu(x) - \left(\int_0^1 x d\mu(x) \right)^2 \tag{11}$$

which requires a numerical integration. Here $f^{(n)}$ is the $n-th$ iterate of the map f.

To proceed with our central subject, the inverse problem, we assume that the first m values of the autocorrelation function are given. This can be due to some experimentally

given data for which a dynamical system should be modelled or due to the fact that a definite behaviour of the correlation function is required in the context of the control of a physical system. As discussed above each set $\{x_p(i), y_p(i), \alpha(i)\}$, $i = 1, 2, .., N$ respecting the constraint of monotony determines uniquely the autocorrelation function $C_{h_f}(n)$ of the corresponding map $f(x)$. One can now ask for the best set $\{x_p(i), y_p(i), \alpha(i), i = 1, ..., N\}$ in the sense that the resulting $C_{h_f}(n)$ possesses the least possible deviation from the given autocorrelation $C(n)$. As a measure for the above-mentioned deviation one can use a χ^2-like cost function. We therefore introduce the following quantity:

$$K[h_f] = \sqrt{\sum_{j=1}^{m} \left(\frac{C_{h_f}(j) - C(j)}{C(j)} \right)^2} \tag{12}$$

The functional $K[h_f]$ is a highly nonlinear function of the parameters $\{x_p(i), y_p(i), \alpha(i)\}$ and we are looking for the global minimum of this function. To perform the minimization of $K[h_f]$ we use a Monte-Carlo (MC) approach based on the Metropolis algorithm [18].

The minimization is performed in several steps, increasing in each step the number of elements $h_{f,i}$ used for the determination of h_f. We start with a lattice consisting of only two points (the origin $x_p(0) = 0$ and the position of the maximum $x_p(1) = x_{\max}$). This means that only one element $h_{f,1}$ is needed for the specification of h_f. The parameters to be fitted in this case are only two: $x_{\max}$ $(x_p(1))$ and the power $\alpha(1)$ determining the behaviour of $h_f(x)$ in the neighbourhood of the origin. The first step ends when the MC minimization has converged to some optimal values for the two fit-parameters. In the second step we use a lattice with three points $x_p(0), x_p(1), x_p(2)$ with $x_p(0) = 0$, $x_p(1) = \frac{x_{\max}}{2}$, $x_p(2) = x_{\max}$. Now we need two elements $h_{f,1}$ and $h_{f,2}$ to determine h_f. We do not keep the values of the old fit-parameters $(x_{\max}, \alpha(1))$ fixed in the second step. Instead we use Gaussian distributed random variables for the choice of the old fit-parameters. The mean values of these Gaussians are the optimum values obtained for these parameters in the previous step and the corresponding widths are taken small enough to allow only weak fluctuations ($\approx 10\%$) around the mean values. Again we perform a MC optimization to obtain optimal values for the two new parameters. This procedure is repeated until the desired convergence is achieved. In each step the lattice size is increased by one point while we include two new parameters.

We use the Metropolis algorithm to find the optimal values for $x_p(i)$ and $\alpha(i+1)$ in each step. For every trial in the i-th turn we assume that $x_p(i)$ and $\alpha(i+1)$ follow a uniform distribution in $(0,1)$ and $(0,\infty)$, respectively (in practice the interval $(0,\infty)$ is replaced by the finite one $(0,c)$ with an upper cutoff $c >> 1$). The annealing is introduced through a thermalized probability distribution of the type: $P = e^{-K[h_f]/T}$ to avoid the trapping into local minima. The parameter T, playing the role of the temperature, is positive and has

to be tuned adiabatically to smaller and smaller values such that the global minimum is reached asymptotically.

We do not keep the values of the old fit-parameters fixed in the following step. Instead we relax this constraint using Gaussian distributed random variables for the choice of the old fit-parameters. The mean values of these Gaussians are the optimum values obtained for these parameters in the previous step and the corresponding widths are taken small enough to allow only weak fluctuations ($\approx 10\%$) around the mean values.

Adding new fit parameters allows us to determine h_f in more and more detail and improves the convergence of the autocorrelation function of the model dynamical system to the given (experimental) autocorrelation function. Indeed, as we shall see below, only a few elements $h_{f,i}$ are required to achieve a rather good convergence. In the next section we give some examples demonstrating how the above-described method can be applied to construct a (piecewise) smooth map simulating a system with given time correlations.

4   Numerical examples and discussion

Before turning to the examples and results of our computational method let us provide some additional aspects concerning the determination of the correlation function. We restrict our investigations to a rather small set of values for the correlation function $C(n)$, more precisely to the set $\{C(0), ..., C(5)\}$. The reason for this restriction is that the exact calculation of the correlation function which has to be accomplished for each single step of the Monte-Carlo scheme is computationally very intensive (see below, the total amount of CPU of our calculations on a powerful workstation was approximately three months). The reliable evaluation of the correlation function is by no means trivial. As demonstrated for example in ref.[19] the results obtained for the correlation function calculated with the trajectories of the dynamical system are, in many cases, not reliable and cannot be improved by going to longer propagation times. Therefore other methods for the calculation of the time correlations are needed. Here we use a numerical approach to eq.(11). It is based on the extraction of the monotony intervals for the n-th iterate. The endpoints of the monotony intervals are given as the preimages of the maximum $x_{\max}$. The integration is then performed for each monotony interval separately. This ensures an accurate although very CPU time consuming evaluation of the correlation function. The latter is related to the exponential growth (2^n) of the number of monotony intervals with increasing n.

We apply now the stochastic method described in the previous section to several examples to demonstrate its capability of providing dynamical systems with prescribed statistical properties. The four different cases we study here can be summarized with respect to their

statistical properties as follows:

- exponentially decaying autocorrelation function and uniform density

- exponentially decaying autocorrelation function and linear density

- oscillatory decaying autocorrelation function and power-law density

- power-law decaying autocorrelation function and uniform density

Figs.(1-4) illustrate the corresponding results of our Monte-Carlo approach. The subfigures (a,c) within each figure show the resulting maps for model I and model II, respectively. Each of the subfigures (b,d) within each figure (1-4) shows both the prescribed data for the autocorrelation signal as well as the result of our optimization approach for model I and II, respectively. In comparing the given autocorrelation data and the results of our optimization scheme we deduce in the following a 'mean relative error' per individual data point $\frac{K[h_f]}{\sqrt{m}}$ of the autocorrelation function.

Since it is not our goal to provide as precise data as possible for the optimized autocorrelation functions but to demonstrate the feasibility of our approach with a good accuracy for a few data points we restrict ourselves to auxiliary functions h_f composed of a few elements $h_{f,i}$. For the case of model I we use (with one exception) four points, i.e. three elements $h_{f,i}$ for the decomposition of the auxiliary function h_f. For model II we use only two points, i.e. one element $h_{f,i}$. Typically a few ten thousand Monte-Carlo steps are performed. The most time consuming part of the algorithm is the calculation of the autocorrelation function which has to be done for each MC step.

First we consider a simple example for which the maps we are looking for are well-known. We use an exponential decay as a prescribed behaviour for the autocorrelation signal while the invariant density of the dynamical process is assumed to be uniform in $[0, 1]$. A corresponding family of maps, the nonsymmetric tent maps, has been studied in some detail in ref.[20]. They are the nonsymmetric tent maps. The resulting maps of our Monte-Carlo optimization are illustrated for model I in fig.1(a) and for model II in fig.1(c). They show only minor differences, i.e. the outcome of approach I is almost the same as for approach II, and both are to a very good approximation an asymmetric tent map. Both models lead also to a very good approximation of the corresponding prescribed correlation function with an error of only 2.2%.

For the second example the two models lead to quantitatively different results. The invariant density is supposed to be linear while the autocorrelation function shows an exponential decay similar to the previous case. The results of the stochastic minimization for this case are illustrated in figure 2. The two obtained dynamical systems (figs.2(a,c)) have a very

different appearance reflecting the fact that our modelling procedure allows for various dynamical systems with the same invariant density and autocorrelation. Let us briefly address the main features of the resulting maps. The map of model I possesses an almost marginal unstable fixed point at the origin and an almost vertical derivative at $x = 1$. Additionally it possesses two obvious points of noncontinuous derivatives located on the left and right branch of the map, respectively. The one on the left branch has a nonvanishing right derivative which, via the function $h_f(x)$, is mapped to a point with almost vertical left derivative in the right branch of $f(x)$. The reader should note that the latter point coincides with the nonzero fixed point of the map ! The map of model II (fig.2(c)) is almost straighlined on its left branch and shows an almost vertical right derivative at the maximum and an almost vertical left derivative at $x = 1$. In general the map of model II is 'smoother' than the map of model I, which is an overall tendency to be observed in any of our examples. It can be viewed as a result of the additional flexibility within model II which allows to independently adapt the left and right derivative in a given interval thereby joining them smoothly together (see the above description for the ansatz of h_f in model II). The prescribed and optimized autocorrelation data for model I are illustrated in subfigure 2(b): They show a deviation of 129% which is predominantly due to the inability of reproducing the single point $C(1)$ within the approach of model I. Although hardly visible in fig.2(b) the prescribed and optimized data coincide very well for $\{C(n), 2 \leq n \leq 5\}$. The optimized correlation function for model II leads to a much better approximation to the prescribed exponential decay and yields an error of only 5.4% (see fig.2(d)).

As a third example we use an invariant density obeying a power law with an exponent $\beta = -0.5$ while the autocorrelation function is chosen to possess a decaying oscillatory behaviour. The corresponding minimization results are shown in fig.3. Both maps (figs.3(a,c)) show an almost horizontal derivative at $x = 1$ and a strong cusp at $x_{\max}$. For the map of model II the left derivative at the maximum is almost vertical. Again the appearance of the map of model II is much smoother compared to the map of model I. Regarding the autocorrelation function we have the opposite situation compared to the previous example. Within model I (fig.3(b)) we obtain a relative error of 22% for the autocorrelation data while model II (fig.3(d) provides an error of 45%. Obviously model I is advantageous in the present case (at least for the present number of grid points chosen for the ansatz of h_f): it nicely reproduces the oscillations of the correlation function.

Finally we study the case that the autocorrelation function decays algebraically with the exponent $\gamma = -2.5$, i.e. we encounter the case of long-range correlations. The invariant density is chosen to be uniform. Both resulting maps (fig.4(a,c)) possess a large but finite derivative at $x = 1$. The map of model I possesses an almost vertical derivative at a single point on its right branch. In contrast to our second example (see above) this point does not

coincide with the fixed point. It is conjectured that this single point with almost vertical derivative is responsible for the observed power law decay of the correlation function (fig4(b)) (see also ref.[21]). Analogously the map of model II (fig.4(c)) shows a very well-pronounced vertical derivative at the cusp. It is interesting to observe that in this case both models I and II provide very satisfactory approximations to the prescribed autocorrelation data (fig4(b,d)), i.e. a relative error of only 13%.

5   Summary

We have introduced a stochastic Monte-Carlo based approach to the inverse problem which uses both the invariant density as well as a finite number of points of the autocorrelation function as prescribed statistical quantities. A key ingredient for this approach was a recently found representation for one-dimensional complete chaotic and single-humped dynamical system in terms of an auxiliary function h_f. This representation is a formal solution of the inverse Frobenius-Perron problem and provides the dynamical system $f(x)$ explicitly as a function of the given measure and the function h_f. h_f therefore reflects the freedom of changing the map without changing its invariant measure (density). In order to quantify the freedom available for the determination of h_f, i.e. to parametrize its functional space, we have introduced two different models which allow to vary h_f extensively. The parameters involved are then used within our stochastic minimization procedure to obtain a correlation function with least deviation from a prescribed autocorrelation signal. Through a number of examples we have demonstrated that our approach possesses an enormous flexibility allowing for a large variety of qualitatively different behaviour of the density and correlation function. To our knowledge there is in general no unique map which belongs to a given density and correlation function. This fact has to be seen in the context of the present investigation as an advantage since it allows for a great flexibility and possible variety with respect to the underlying dynamical systems.

We would like to mention that the above-discussed prescribed behaviour of the autocorrelation data (exponential, oscillating, power law decay) is, strictly speaking, enforced only for the first five points included in our Monte-Carlo optimization. In principle it is imaginable that this behaviour represents a transient and the asymptotics of the correlation function might show a different behaviour. To determine exact asymptotic properties of certain dynamical systems is however not the issue of the present paper. Our goal is to extend the inverse problem by including the correlation function in terms of a few (experimentally) available data points, thereby enabling us to design a dynamical system with desired statistical properties. Furthermore our approach might be suggestive in terms of influencing or

controlling dynamical systems [9, 10] in a certain way.

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FIGURE CAPTIONS

Figure 1: The stochastic minimization results for a dynamical system with uniform invariant density and exponentially decaying autocorrelation function. (a) The resulting map using model I. (b) The autocorrelation function for the map of model I (solid line) and the corresponding prescribed data (full circles). (c) The resulting map using model II. (d) The autocorrelation function for the map of model II (solid line) and the corresponding prescribed data (full circles).

Figure 2: Same as in figure 1 but for prescribed linear invariant density and exponentially decaying autocorrelation function.

Figure 3: Same as in figure 1 but for prescribed power-law invariant density ($\beta = -0.5$) and oscillatory decaying autocorrelation function.

Figure 4: Same as in figure 1 but for prescribed uniform invariant density and power-law decaying autocorrelation function (with exponent -2.5).
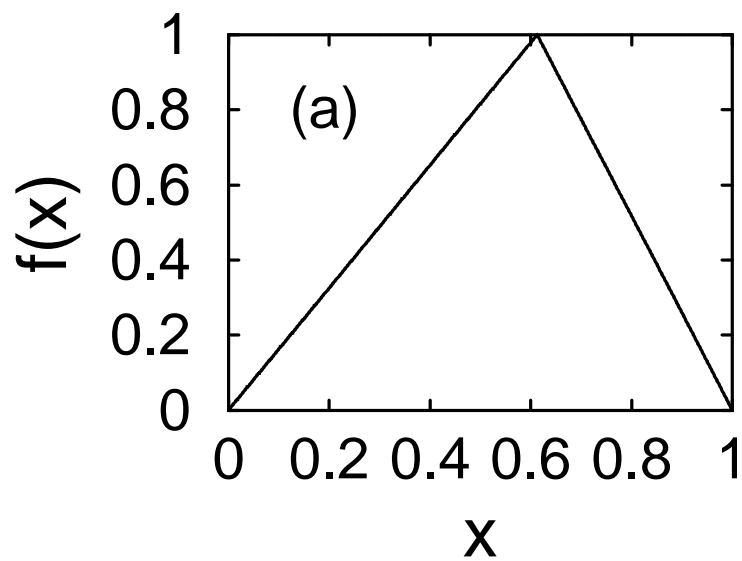
Fig. 1

Fig.2

Fig.3

Fig.4